THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2023-2024 Term 1
Homework Assignment 3
Due Date: 12 October, 2023 (Thursday)

General Regulations

- All assignments will be submitted and graded on CUHK Blackboard. You can view your grades and submit regrade requests here as well.

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win $2 for each black ball selected and we lose $1 for each white ball selected. Let X denote our winnings.

   (a) What are the possible values of X, and what are the probabilities associated with each value?

   (b) If we play the game 100 times, and to play every time we have to pay $2 as table money, what is the amount we should expect to get?

   (c) Is the game fair? Explain.

2. Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman. (For instance, $X = 1$ if the top-ranked person is female.) Find $P\{X = i\}$ or $P(X = i)$, $i = 1, 2, 3, \cdots, 8, 9, 10$.

   In what follows, we use the round parentheses instead of the curly parentheses.

3. A total of 4 buses carrying 148 students from a school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X be the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y be the number of students on her bus. Determine $E[X]$, $E[Y]$, $\text{Var}(X)$ and $\text{Var}(Y)$.

4. Two coins are to be flipped. The first coin will land on heads with probability .6 and the second with probability .7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

   (a) Find $P(X = 1)$.

   (b) Determine $E[X]$.

5. When coin 1 is flipped, it lands on heads with probability .4; when coin 2 is flipped, it lands on heads with probability .7. One of these coins is randomly chosen and flipped 10 times.

   (a) What is the probability that the coin lands on heads for exactly 7 of the 10 flips?

   (b) Given that the first of these ten flips lands on heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

6. An interviewer is given a list of people she can interview. If the interviewer needs to interview 5 people, and if each person (independently) agrees to be interviewed with probability 2/3,

   (a) For part (a), what is the probability that her list of people will enable her to obtain her necessary number of interviews if the list consists of (a) 5 people and (b) 8 people?

(b) For part (b), what is the probability that the interviewer will speak to exactly (c) 6 people and (d) 7 people on the list?

Note that there are four subproblems in this question.

7. Let X be such that
$$P(X = 1) = p = 1 - P(X = -1)$$
Find $c \neq 1$ such that $E[c^X] = 1$.

8. Let X be a binomial random variable with parameters n and p. Show that
$$E\left[\frac{1}{X+1}\right] = \frac{1 - (1-p)^{n+1}}{(n+1)p}.$$

9. Let X be a Poisson random variable with parameter λ. What value of λ maximizes $P(X = k)$, $k \geq 0$?

10. Show that X is a Poisson random variable with parameter λ, then
$$E[X^n] = \lambda E\left[(X+1)^{n-1}\right]$$

Now use this result to compute $E[X^3]$.